2016

MATHEMATICS

(Major)

Paper : 6.4

(Discrete Mathematics)

Full Marks : 60

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Answer the following as directed : 1×7=7

(a) State Peano's axioms.

(b) Show that the integer 1999 is prime.

(c) If $a|bc$ and $\gcd(a, b) = 1$, $a, b, c \in \mathbb{Z}$, then show that $a|c$.

(d) State Chinese remainder theorem.

(e) $5^{10} - 3^{10}$ is divisible by 11. Justify whether it is true or false.

(f) What is the remainder when 7^{30} is divided by 4?

(g) If $\gcd(x, y) = 1$ and $xy = d^2$, then x and y are also squares. Justify whether it is true or false.

2. Answer the following questions : $2 \times 4 = 8$

(a) If a, b, c are positive integers such that $\gcd(a, b) = 1 = \gcd(a, c)$, then prove that $\gcd(a, bc) = 1$.

(b) Show that every square number is of the form $5k - 1$, $5k$, $5k + 1$, where k is some positive integers.

(c) Find all solutions of the Diophantine equation $3x + 2y = 6$.

(d) If p is a prime of the form $4k + 1$, show that there exists a solution in integers x, y, m of $x^2 + y^2 = mp$, where $0 < m < p$.

3. Answer the following questions : $5 \times 3 = 15$

(a) Prove that every integer $n \geq 2$ has a prime factor.

Or

Prove that there are infinitely many primes of the form $4n + 3$.

(b) If p is a prime, then show that

$$2\{(p-3)!\} + 1 \equiv 0 \pmod{p}$$

Or

Solve the congruence :

$$72x \equiv 18 \pmod{42}$$

(c) Prove that if an odd prime number p can be written as a sum of two squares, then $p \equiv 1 \pmod{4}$.

4. (a) Answer *either* (i) *or* (ii) : 10

(i) (1) If n is a positive integer and $\gcd(a, n) = 1$, then show that

$$a^{\phi(n)} \equiv 1 \pmod{n}$$ 5

(2) If $n = p_1^{k_1} p_2^{k_2} \ldots p_t^{kt}$ is the prime factorization of $n > 1$, then show that

$$\sigma(n) = \prod_{i=1}^{t} \frac{p_i^{k_i+1} - 1}{p_i - 1}$$

Hence show that σ is a multiplicative function. 3+2=5

(ii) (1) Show that if p and q are distinct primes, then

$$p^{q-1} + q^{p-1} \equiv 1 \pmod{pq}$$ 7

(2) Prove that if 5 divides n, then

$$\phi(5n) = 5\phi(n)$$ 3

(b) Answer *either (i) or (ii)* : 10

(i) (1) If A is a tautology and $A \to B$, then show that B is also a tautology. Examine if the following are adequate systems of connectives : 5

(i) $(\sim, \wedge)$

(ii) $(\vee, \to)$

(2) Let p stand for 'π is a rational number', q for 'a triangle has two sides' and r for 'the earth revolves around the sun'. Then find the truth value of the following statement forms : 5

(i) $p \vee (\sim q \wedge r)$

(ii) $(p \wedge q) \to r$

(iii) $((\sim p) \vee (\sim q)) \leftrightarrow (q \to r)$

(ii) Prove that the collection $(\sim, \wedge, \vee)$ is an adequate system of connectives. 10

(c) Answer *either (i) or (ii)* : 10

(i) (1) What is meant by a complete disjunctive normal form (DNF)? Give one example of a complete DNF. Show that a complete DNF is identically 1. 5

(2) Express the following Boolean expression in disjunctive normal form and conjunctive normal form in the variables present in the expression

$$(xy' + xz)' + x'$$ 5

(ii) A committee consisting of three members approves any proposal by majority vote. Each member can approve a proposal by pressing a button attached to their seats. Design as simple a circuit as you can which will allow current to pass when and only when a proposal is approved. 10

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