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# MATHEMATICS

( Major )

Paper : 3.2

**( Linear Algebra and Vector )**

*Full Marks* : 80

*Time* : 3 hours

*The figures in the margin indicate full marks for the questions*

## GROUP—A

**( Linear Algebra )**

( *Marks* : 40 )

**1.** Answer the following as directed : 1×7=7

(a) Let $U = \{(a, b, c) : a = b = c\}$ is a subset in $\mathbb{R}^3$. Determine whether or not $U$ is a subspace of $\mathbb{R}^3$.

(b) If $P_n(t)$ be the vector space of all polynomials of degree $\leq n$, then dim $P_n(t)$ is

(i) $n-1$ (ii) $n$

(iii) $n+1$ (iv) $n^2$

(Choose the correct option)

(c) The set $\mathbb{R}$ of all real numbers is a vector space over the field $\mathbb{Q}$ of rational numbers. Examine whether or not the set $\{1, \sqrt{2}\}$ of vectors in $\mathbb{R}$ is linearly independent.

(d) Let $v = (1, 2, 3)$ and scalar $k = -3$, show that $T : R^3 \to R^2$ defined by $T(x, y) = (|x|, y + z)$ is not linear.

(e) If $T$ is a linear operator, then the following are equivalent :

(i) A scalar $\lambda$ is an eigenvalue of $T$

(ii) The linear operator $\lambda I - T$ is singular

(Write true or false)

(f) Let $V$ be vector space over the field $F$ and let $T$ be a linear operator on $V$. Define the characteristic space associated with a characteristic value of $T$.

(g) Prove that for a linear operator (matrix) $T$, the scalar 0 is an eigenvalue of $T$ if and only if $T$ is singular.





2. Answer the following questions : 2×4=8

(a) Express $v = (2, -5, 3)$ in $\mathbb{R}^3$ as a linear combination of the vectors

$$u_1 = (1, -3, 2),\ u_2 = (2, -4, -1),\ u_3 = (1, -5, 7)$$

(b) Let $\mathbb{R}$ be the field of real numbers and $V$ be the space of all functional from $\mathbb{R}$ into $\mathbb{R}$ which are continuous. Define $T$ by

$$(Tf)(x) = \int_0^x f(t)\, dt$$

Show that $T$ is a linear transformation from $V$ into $V$.

(c) Consider the two bases of the vector space $\mathbb{R}^2(\mathbb{R})$ :

$$B_1 = \{(1, 2), (3, 5)\} \text{ and } B_2 = \{(1, -1), (1, -2)\}$$

Find the change-of-basis matrix $M$ from $B_1$ to the 'new' basis $B_2$.

(d) Using Cayley-Hamilton theorem, compute the inverse of

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}$$

**3.** Answer any *one* part : 5

(a) Let $V$ and $W$ be vector spaces over the same field $F$ and $T$ be a linear transformation from $V$ into $W$. Show that if $V$ is finite dimensional, then

$$\text{rank}(T) + \text{nullity}(T) = \dim V$$

(b) Let $V$ and $W$ be vector spaces over the same field $F$ and the linear mapping $T: V \to W$ is one-to-one and onto. Show that the inverse map $T^{-1}: W \to V$ is also linear.

**4.** Answer the following questions : 10×2=20

(a) When a vector space is said to be finitely generated? If $V$ is a finitely generated vector space over a field $F$, prove that $V$ has a finite basis and any two bases of $V$ have same number of vectors.

*Or*

Suppose $V$ is finite dimensional vector space over a field $F$ and $U$ is a subspace of $V$. Prove that there is a subspace $W$ of $V$ such that $V = U \oplus W$.





(b) State and prove Cayley-Hamilton theorem for the characteristic polynomial $f$ of a linear operator $T$ on a finite dimensional vector space $V$.

*Or*

Let $P$ be the operator on $\mathbb{R}^2$ which projects each vector onto the $x$-axis, parallel to the $y$-axis, $P(x, y) = (x, 0)$. Show that $P$ is linear. What is the minimal polynomial for $P$?

## GROUP—B

## ( Vector )

( *Marks* : 40 )

**5.** Answer the following : 1×3=3

(a) Find the constant $p$ such that the vectors

$$\vec{a} = 2\vec{i} - \vec{j} + \vec{k},\ \vec{b} = \vec{i} + 2\vec{j} - 3\vec{k},\ \vec{c} = 3\vec{i} + p\vec{j} + 5\vec{k}$$

are coplanar.

(b) Prove that

$$(\vec{a} \times \vec{b}) \times (\vec{a} \times \vec{c}) \cdot \vec{d} = (\vec{a} \cdot \vec{d})[\vec{a}\ \vec{b}\ \vec{c}]$$

(c) If $\vec{a}$ and $\vec{b}$ lie in a plane normal to the plane containing $\vec{c}$ and $\vec{d}$, show that

$$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = 0$$

6. If $\vec{a}_1, \vec{b}_1, \vec{c}_1$ and $\vec{a}_2, \vec{b}_2, \vec{c}_2$ are reciprocal system of vectors, prove that

$$\vec{a}_1 \times \vec{a}_2 + \vec{b}_1 \times \vec{b}_2 + \vec{c}_1 \times \vec{c}_2 = \vec{0}$$ 2

7. Answer the following questions : 5×3=15

(a) If $\vec{a}, \vec{b}, \vec{c}$ are the position vectors of $A$, $B$, $C$, prove that

$$\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}$$

is a vector perpendicular to the plane $ABC$.

(b) (i) If $\frac{d\vec{p}}{dt} = \vec{u} \times \vec{p}$ and $\frac{d\vec{q}}{dt} = \vec{u} \times \vec{q}$

show that

$$\frac{d}{dt}(\vec{p} \times \vec{q}) = \vec{u} \times (\vec{p} \times \vec{q})$$

(ii) If $\vec{w} = 5t^2\vec{i} + t\vec{j} - t^3\vec{k}$, evaluate

$$\frac{d}{dt}(\vec{w} \cdot \vec{w})$$

3+2=5

(c) Prove that

$$\nabla \times (\nabla \times \vec{F}) = \nabla(\nabla \cdot \vec{F}) - \nabla^2 \vec{F}$$





8. Answer the following questions : 10×2=20

(a) If $\vec{c}$ is a constant vector, prove that

$$\text{div}\,(r^n (\vec{c} \times \vec{r})) = 0,$$

where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.

*Or*

Prove that the necessary and sufficient condition for a vector $\vec{v}(t)$ to have a constant direction is

$$\vec{v} \times \frac{d\vec{v}}{dt} = \vec{0}$$

(b) Evaluate

$$\int_C \vec{F} \cdot d\vec{r}$$

where $\vec{F}(x, y, z) = yz\vec{i} - xz\vec{k}$ and $C$ is the line segment from $(-1, 2, 0)$ and $(3, 0, 1)$.

*Or*

Find the work done when a force

$$\vec{F} = (x^2 - y^2 + x)\vec{i} - (2xy + y)\vec{j}$$

moves a particle in $xy$-plane from $(0, 0)$ to $(1, 1)$ along the parabola $y^2 = x$.

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