

Total number of printed pages-8

3 (Sem-6/CBCS) MAT HC 1

2022

MATHEMATICS

(Honours)

Paper : MAT-HC-6016

(Complex Analysis)

Full Marks : 60

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer **any seven** questions from the following: 1×7=7

(a) If c is any n th root of unity other than unity itself, then value of

$1 + c + c^2 + \dots + c^{n-1}$ is

(i) $2n\pi$

(ii) 0

(iii) -1

(iv) None of the above

(Choose the correct answer)

Contd. 118

(b) The square roots of $2i$ is

(i) $\pm (1+i)$

(ii) $\pm (1-i)$

(iii) $\pm \frac{1}{\sqrt{2}} (1-i\sqrt{2})$

(iv) None of the above

(Choose the correct answer)

(c) A composition of continuous function is

(i) discontinuous

(ii) itself continuous

(iii) pointwise continuous

(iv) None of the above

(Choose the correct answer)

(d) The value of $\text{Log}(-ei)$ is

(i) $\frac{\pi}{2} - i$

(ii) i

(iii) $1 - \frac{\pi}{2}i$

(iv) None of the above

(Choose the correct answer)

(e) The power expression of $\cos z$ is

(i) $\frac{e^z + e^{-z}}{2}$

(ii) $\frac{e^{iz} + e^{-iz}}{2}$

(iii) $\frac{e^{iz} + e^{-iz}}{2i}$

(iv) None of the above

(Choose the correct answer)

(f) The Cauchy-Riemann equation for analytic function $f(z) = u + iv$ is

(i) $u_x = v_y, u_y = -v_x$

(ii) $u_x = -v_y, u_y = v_x$

(iii) $u_{xx} + v_{yy} = 0$

(iv) None of the above

(Choose the correct answer)

(g) If $w(t) = u(t) + iv(t)$, then $\frac{d}{dt}[w(t)]^2$ is equal to

(i) $2[u(t) + iv(t)]$

(ii) $2w'(t)$

(iii) $2w(t)w'(t)$

(iv) None of the above

(Choose the correct answer)

- (h) What is Laplace's equation?
- (i) What is extended complex plane?
- (j) What is Jordan arc?

2. Answer **any four** questions from the following: 2×4=8

(a) Write principal value of $\arg\left(\frac{i}{-1-i}\right)$.

(b) If $f(z) = x^2 + y^2 - 2y + i(2x - 2xy)$, where $z = x + iy$, then write $f(z)$ in terms of z .

(c) Use definition to show that

$$\lim_{z \rightarrow z_0} \bar{z} = \bar{z}_0.$$

(d) Find the singular point of

$$f(z) = \frac{z^2 + 3}{(z + 1)(z^2 + 5)}.$$

(e) If $f'(z) = 0$ everywhere in a domain D , then prove that $f(z)$ must be constant throughout D .

(f) Evaluate $f'(z)$ from definition, where

$$f(z) = \frac{1}{z}.$$

(g) If $f(z) = \frac{z}{\bar{z}}$, find $\lim_{z \rightarrow 0} f(z)$, if it exists.

(h) Write the function $f(z) = z + \frac{1}{z}$ ($z \neq 0$)

in the form $f(z) = u(r, \theta) + iv(r, \theta)$.

3. Answer **any three** questions from the following : 5×3=15

(a) If z_1 and z_2 are complex numbers, then show that

$$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2.$$

(b) Show that $\exp. (2 \pm 3\pi i) = -e^2$.

(c) Sketch the set $|z - 2 + i| \leq 1$ and determine its domain.

(d) Let C be the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$, that lies in the 1st quadrant, then show that

$$\left| \int_C \frac{z-2}{z^2+1} dz \right| \leq \frac{4\pi}{15}$$

(e) Evaluate $\int_C \frac{dz}{z}$, where C is the top half of the circle $|z|=1$ from $z=1$ to $z=-1$.

(f) If $f(z) = e^z$, then show that it is an analytic function.

(g) If $f(z) = \frac{z+2}{z}$ and C is the semi circle $z = 2e^{i\theta}$, $(0 \leq \theta \leq \pi)$, then evaluate $\int_C f(z) dz$.

(h) Find all values of z such that $e^z = -2$.

4. Answer **any three** questions from the following : 10×3=30

(a) State and prove Cauchy-Riemann equations of an analytic function in polar form.

(b) Suppose that

$$f(z) = u(x, y) + iv(x, y), \quad (z = x + iy)$$

and $z_0 = x_0 + iy_0$, $w_0 = u_0 + iv_0$, then

prove that if $\lim_{(x, y) \rightarrow (x_0, y_0)} u(x, y) = u_0$

and $\lim_{(x, y) \rightarrow (x_0, y_0)} v(x, y) = v_0$ then

$\lim_{z \rightarrow z_0} f(z) = w_0$ and conversely.

- (c) If the function $f(z) = u(x, y) + iv(x, y)$ is defined by means of the equation

$$f(z) = \begin{cases} \frac{\bar{z}^e}{z}, & \text{when } z \neq 0 \\ 0, & \text{when } z = 0, \end{cases}$$

then prove that its real and imaginary parts satisfies Cauchy-Riemann equations at $z = 0$. Also show that $f'(0)$ fails to exist.

- (d) If the function

$f(z) = u(x, y) + iv(x, y)$ and its conjugate $\bar{f}(z) = u(x, y) - iv(x, y)$ are both analytic in a domain D , then show that $f(z)$ must be constant throughout D .

- (e) If f be analytic everywhere inside and on a simply closed contour C , taken in the positive sense and z_0 is any point interior to C , then prove that

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz.$$

- (f) State and prove Liouville's theorem.

(g) Suppose that a function f is analytic throughout a disc $|z - z_0| < R_0$ centred at z_0 and with radius R_0 . Then prove that $f(z)$ has the power series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad (|z - z_0| < R_0)$$

$$\text{where } a_n = \frac{f^n(z_0)}{n!}, \quad (n = 0, 1, 2, \dots)$$

(h) State and prove Laurent's theorem.